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Enlarged Geometries of Gauge Bundles

R. Aldrovandi[1] and A. L. Barbosa[1]

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The geometrical picture of gauge theories must be enlarged when a gauge potential ceases to behave like a connection, as it does in electroweak interactions. When the gauge group has dimension four, the vector space isomorphism between spacetime and the gauge algebra is realized by a tetrad-like field. The object measuring the deviation from a strict bundle structure has the formal behavior of a spacetime connection, of which the deformed gauge field strength is the torsion. A generalized derivative emerges in terms of which the two Bianchi identities are formally recovered. Effects of gravitational type turn up. The dynamical equations obtained correspond to a broken gauge model on a curved spacetime.

1. INTRODUCTION

Differential geometry, in its modern fiber-bundle language, provides the mathematical background for theories describing the known fundamental interactions. The bundle of frames stands behind general relativity, while other principal bundles, built up with the respective gauge groups, give a clear picture of the kinematic setup behind electroweak and strong interactions (Trautman, 1970; Wu and Yang, 1975; Daniel and Viallet, 1980). The picture is commonplace: geometry supplies the stage set on which Lagrangians of phenomenological origin rule over dynamics. Dynamics confers different characters to gravitation, whose Lagrangian is of first order in the curvature, and to the other interactions, whose Lagrangians are of second order in the curvature. But in all cases it is a curvature which appears, and curvature is a quantity derived from a connection. The metric keeps the central role in gravitation, but the basic fields in the other cases are gauge potentials, that

[1] Laboratoire de Gravitation et Cosmologie Relativistes, Université Pierre et Marie Curie, CNRS/ESA 7065, Cedex 05, 75252 Paris, France.

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is, connections. A splendid experimental record favors the existing theories and justifies the belief that much of their content is of perennial value.

There are, however, some cloudy spots in this sunny landscape. There are too many arbitrary constants and a obstinate lack of unity with respect to general principles. Gravitation alone is universal, can be locally simulated by a moving frame, has a problematic energy and is power-counting nonrenormalizable. Some of the mediating bosons are massless and have problematic charges. Other are massive and have well-defined charges. And there is the question of the meaning to be attributed to spontaneous symmetry breakdown. The presence of a remnant scalar field adds to the difficulties (Gaillard *et al.*, 1999). This blending of experimental success and theoretical bafflement suggests that, though the gauge principle is promised an important role in an eventual final theory, the simple, direct gauge prescription will not have the last say as it stands. In the search for a more comprehensive framework, string theory, with its ultimate goal of explaining, in principle, "everything," is the dominating trend.

We present here a few more steps toward another proposal (Aldrovandi, 1995), which starts from gauge theories and looks for the minimal modifications necessary to shed light on at least some of these difficulties. It takes into account two initial clues. The first is supported by all the experimental evidence and is concerned with the peculiar behavior of the electroweak gauge potentials. The gauge potentials appearing in chromodynamics and isolated electromagnetism, as well as the Christoffel symbols in gravitation, behave strictly as connections, but the vector fields describing real particles in electroweak theory do not. The theory does start with a connection-behaving gauge potential, but then spontaneously breaks the symmetry by introducing an external field. The final combinations, representing the physical fields, do not transform as connections. This leads to the second clue, more mathematical in nature: when a gauge field ceases to behave like a connection, the whole geometric picture provided by the underlying bundle is blurred. What happens to the bundle picture when a connection, or part of it, adopts an abnormal behavior?

The connection adjoint behavior is essential to the bundle picture. On the bundle tangent spaces it is reflected in the commutators of the vector fields coming from the base manifold (external space, spacetime) and from the structure (internal, gauge space) group. Vector fields are derivatives, and a connection allows mixing internal and external vectors to produce more general, covariant derivatives while preserving the bundle makeup. This preservation comes from the connection adjoint behavior. Any deviation from that behavior changes the whole picture, and the electroweak physical fields do deviate. Some encouraging results were obtained in which an abnormal behavior of a gauge potential was shown to engender fields strongly suggestive of linear connections with their curvatures and torsions (Aldrovandi, 1991a), hinting thereby at a relationship with gravitation. We intend here to present some new results on the subject, valid when the gauge group has, as in the Weinberg–Salam theory, dimension 4.

We start (Section 2) with a formal compact on Lie algebra extensions. In this section we also examine the behavior of the Lie algebra of fields on a manifold under changes of basis. To alleviate notation we shall, as a rule, omit projections, their differentials, and corresponding pullbacks. In Section 3 we introduce an enlarged concept of change of basis in the principal fiber bundle and we apply it successively to the simplest conceivable geometric configuration and obtain three kinds of commutation relations: those of a gauge theory, those of an extended gauge theory, and those of a gravitational model. Noncovariant derivatives, akin to those appearing in electroweak theory, turn up naturally in the extended formalism. In Section 4 we begin discussing which aspects of the geometric picture can be retained in the presence of anomalous connections. When the base manifold and the gauge group have the same dimension, as is the case involving spacetime and the electroweak theory, a tetrad-like field is naturally introduced to represent the isomorphism of the underlying vector spaces of the tangent field algebras. The object measuring the breaking of the bundle structure acquires the aspect of an external, linear connection, which preserves the metric defined by the tetrad and is endowed with curvature and torsion. Thus, the same *objects* of usual geometry are found and strongly suggest a relation to gravitation. Section 5 shows that such objects have the expected geometrical properties and lead to reasonable dynamical equations. Many results previously found for the translation group (Aldrovandi, 1991b) are extended to the non-Abelian case. It should be emphasized that, due to the presence of noncovariant objects, even the most trivial formulas of tensor calculus must be reworked from the start. Some of them survive, others are modified. A very general derivative shows up involving simultaneously gauge and "gravitational" aspects. Dynamics for the gauge sector can be obtained by assuming the persistence of the duality symmetry and, for the gravity sector, by a procedure analogous to that used in general relativity. The final section sums up the results and the many unsolved problems.

2. EXTENSIONS OF TANGENT ALGEBRAS

We shall find it necessary to call attention to a certain number of elementary facts, and introduce notation through an overview of well-known notions. A Lie algebra is a vector space on which a binary internal operation is defined which is antisymmetric and satisfies the Jacobi identity. The operation will be indicated by the commutator, and the algebra whose underlying vector space is V will be denoted V'. For simplicity, the same notation will be used for a Lie group G and its Lie algebra G'. The algebra is characterized by the operation table written in a vector basis $\{Y_\alpha\}$ of members, $[Y_\alpha, Y_\beta]_V = f^\gamma{}_{\alpha\beta}\, Y_\gamma$. The numbers $f^\gamma{}_{\alpha\beta}$ are the structure constants of V'. We shall sometimes indicate the algebra by one of its bases, as in $V' = \{Y_\alpha\}$.

In order to discuss the extension of a Lie algebra L' by another Lie algebra V', notice that the direct sum $E = L \oplus V$ of the underlying vector spaces L and V is always defined. To extend L' by V' means, in general terms, to give an answer to the following question: when and how can we combine L' and V' to build another Lie algebra E' with underlying vector space $L \oplus V$? In the generic case, many answers are possible, provided L' has a representation acting on V'. Two main points should be specified: (i) the insertion of the algebras in the enlarged space E and (ii) the relationship between the algebras after the insertion.

We shall be interested in extensions involving the algebras of vector fields on differentiable manifolds. The pattern introduced below is closely related to that present on the total manifold P of a principal bundle (Kobayashi and Nomizu, 1963).

Let P be a differentiable manifold. It will have a tangent space T_pP at each point $p \in P$. A vector field X is a differentiable choice of a vector X_p at each T_pP. In general, making such a choice is only possible locally, that is, on an open neighborhood of each point p. For that reason all the discussion which follows will be purely local in character. If the manifold is C^∞, X will act on a space $\mathbf{R}(P)$ of infinitely differentiable real functions on P.

The set of all vector fields on P constitutes an infinite Lie algebra $\Xi(P)$. Consider a Lie group whose Lie algebra G' has generators J_μ satisfying the commutation rules

$$[J_\mu, J_\nu] = f^\lambda{}_{\mu\nu}\, J_\lambda \tag{1}$$

When G acts on P as a transformation group, there is a representation ρ of its generators by fields on P. This means (Aldrovandi and Pereira, 1995) that ρ chooses, for each J_μ, a *representative* field $Y_\mu \in \Xi(P)$:

$$\begin{aligned} \rho: &\quad G' \to \Xi(P) \\ \rho: &\quad J_\mu \to Y_\mu = \rho(J_\mu) \end{aligned} \tag{2}$$

The representation ρ will be a *linear representation* when the representative fields have the same commutation rules as the fields they represent:

$$[Y_\mu, Y_\nu]_{\Xi(P)} = f^\lambda{}_{\mu\nu}\, Y_\lambda \tag{3}$$

Suppose that a first representative algebra $L' = \{Y_\mu\}$ is given around a point

p on P, with a number $d < \dim P$ of generators. Consider also a linear representation, also around p, of another algebra V', locally given by a number $n = \dim P - d$ of fields X_a with commutations

$$[X_a, X_b]_{\Xi(P)} = f^c{}_{ab} X_c \tag{4}$$

If all the involved fields Y_μ and X_a are linearly independent, the set $\{Y_\mu, X_a\}$ constitutes a local basis around p. Notice that, once an algebra is represented by vectors at a point $p \in P$, its structure *constants* can become point dependent (structure *coefficients*) when these vectors are extended into vector fields around p. As a last basic assumption, suppose the commutation table in that basis has the form

$$\begin{aligned} [Y_\mu, Y_\nu]_{\Xi(P]} &= f^\lambda{}_{\mu\nu} Y_\lambda - \beta^a{}_{\mu\nu} X_a \\ [Y_\mu, X_a]_{\Xi(P)} &= C^b{}_{\mu a} X_b \\ [X_a, X_b]_{\Xi(P)} &= f^c{}_{ab} X_c \end{aligned} \tag{5}$$

$\beta_{\mu\nu}$ is a 2-form with values in the V' sector. It characterizes the deviation from the linearity (3) of the algebra $\{Y_\mu\}$ caused by its association with the algebra $\{X_a\}$. The latter, by the above relations, is unaffected: it is simply included in E', and its structure coefficients remain constant:

$$[X_a, X_b]_{\Xi(P)} = [X_a, X_b]_V = f^c{}_{ab} X_c$$

The middle expression in (5) says that the result of any action of L' on V' stays in V'. For each fixed μ, the field Y_μ is represented on the X_a by the matrix C_μ whose entries are the coefficients $C^b{}_{\mu a}$. The algebra E' specified by (5) is an extension of the representative field algebra of L' by the representative field algebra of V'.

An extension is *trivial* when there is no departure from linearity, that is, when $\beta^a{}_{\mu\nu} = 0$. The extension is a *direct product* when the fields Y_μ act on the X_a by the null representation, that is, when $C^b{}_{\mu a} = 0$. This will be a necessary (but not sufficient) condition for the geometry of gauge theories.

The compound so obtained depends, thus, on the pair $(C^b{}_{\mu a}, \beta^a{}_{\mu\nu})$. The extended algebra should be a Lie algebra, so that we impose the Jacobi identities on the fields obeying (5). Three conditions result which must be respected by any pair $(C^b{}_{\mu a}, \beta^a{}_{\mu\nu})$:

$$\begin{aligned} &Y_\mu(\beta^a{}_{\nu\sigma}) + Y_\sigma(\beta^a{}_{\mu\nu}) + Y_\nu(\beta^a{}_{\sigma\mu}) + C^a{}_{\nu c}\beta^c{}_{\sigma\mu} + C^a{}_{\sigma c}\beta^c{}_{\mu\nu} + C^a{}_{\mu c}\beta^c{}_{\nu\sigma} \\ &\quad + f^\rho{}_{\mu\nu}\beta^a{}_{\sigma\rho} + f^\rho{}_{\sigma\mu}\beta^a{}_{\nu\rho} + f^\rho{}_{\nu\sigma}\beta^a{}_{\mu\rho} = 0 \end{aligned} \tag{6}$$

$$\begin{aligned} &Y_\mu(C^a{}_{\nu b}) - Y_\nu(C^a{}_{\mu b}) + C^a{}_{\mu c}C^c{}_{\nu b} - C^a{}_{\nu c}C^c{}_{\mu b} - f^\rho{}_{\mu\nu}C^a{}_{\rho b} \\ &\quad - X_b(\beta^a{}_{\mu\nu}) - \beta^c{}_{\mu\nu} f^a{}_{bc} = 0 \end{aligned} \tag{7}$$

$$X_a(C^c{}_{\mu b}) - X_b(C^c{}_{\mu a}) - C^d{}_{\mu a} f^c{}_{bd} + C^c{}_{\mu d} f^d{}_{ba} + C^d{}_{\mu b} f^c{}_{ad} = 0 \tag{8}$$

An extension is *central* when $\beta^c{}_{\mu\nu} X_c$ has all its elements in the center of the algebra V'. In particular, it follows from (7) that every direct product ($C^a{}_{\mu b} = 0$) is a central extension. In effect, in that case

$$[X_b, \beta^a{}_{\mu\nu} X_a]_{\Xi(P)} = \{X_b(\beta^c{}_{\mu\nu}) + f^c{}_{ba}\beta^a{}_{\mu\nu}\} X_c = 0 \tag{9}$$

Let us examine what happens to the above commutation tables under a change of basis. Starting from the basis $\{Y_\mu, X_\alpha\}$ on the whole manifold P, we introduce the particular transformationss

$$Y'_\mu = Y_\mu - \alpha^c{}_\mu(x) X_c \tag{10}$$

where the $\alpha^a{}_\mu$ are point-dependent objects on P. In the applications we have in mind P will be the total space of a bundle with spacetime as base manifold. The fields Y_μ will represent translations on spacetime, so that the coefficients $f^\rho{}_{\mu\nu}$ are a mere signal of anholonomy. We shall take for $\{Y_\mu\}$ a holonomic basis, so that $f^\rho{}_{\mu\nu} = 0$ in (5)–(7). Notice that the nonlinearity indicator $\beta^a{}_{\mu\nu}$ is not an anholonomy coefficient for $\{Y_\mu\}$, as it points toward other directions in the algebra. Notice that we consider (10) as a change of basis on the whole local algebra of vector fields on P. The new set of commutation relations is (we also drop the index $\Xi(P)$ from now on)

$$[Y'_\mu, Y'_\nu] = -\beta'^a{}_{\mu\nu} X_a \tag{11}$$

$$[Y'_\mu, X_a] = C'^b{}_{\mu a} X_b \tag{12}$$

$$[X_a, X_b] = f^c{}_{ab} X_c \tag{13}$$

with new coefficients given by

$$C'^b{}_{\mu a} = C^b{}_{\mu a} - \alpha^c{}_\mu f^b{}_{ca} + X_a(\alpha^b{}_\mu) \tag{14}$$

$$\beta'^a{}_{\mu\nu} = \beta^a{}_{\mu\nu} + K^a{}_{\mu\nu} \tag{15}$$

and

$$\begin{aligned} K^a{}_{\mu\nu} = {} & Y'_\mu \alpha^a{}_\nu - Y'_\nu \alpha^a{}_\mu + \alpha^b{}_\mu X_b(\alpha^a{}_\nu) - \alpha^b{}_\nu X_b(\alpha^a{}_\mu) \\ & + \alpha^b{}_\nu C'^a{}_{\mu b} - \alpha^b{}_\mu C'^a{}_{\nu b} + f^a{}_{bc}\alpha^b{}_\mu \alpha^c{}_\nu \end{aligned} \tag{16}$$

Relations (14)–(16) are such that the forms of the Jacobi identities are preserved under (10). This is important because, as we shall see later, the field equations will come from Jacobi identities.

3. CHANGES OF BASIS

The simple scheme of basis transformation presented above can, if we start from a trivial initial algebra, engender three types of algebra: that of a gauge theory, the extension given above or the forthcoming extended algebra of Section 4, and the algebra corresponding to a gravitational model. Assuming the validity of the duality prescription applied to the Bianchi identities, we can also obtain the corresponding dynamics of each theory.

We shall take as starting field configuration that appearing on a fiber bundle whose structure group G has Lie algebra $G' = \{X_a\}$ and whose base manifold is spacetime represented by the trivial holonomic basis $\{\partial_\mu\}$. The set of commutation relations is

$$\begin{aligned} [\partial_\mu, \partial_\nu] &= 0 \\ [X_a, \partial_\mu] &= 0 \\ [X_a, X_b] &= f^c{}_{ab} X_c \end{aligned} \tag{17}$$

It represents a trivial and direct-product extension of the translation algebra by G', or vice versa. Physically, it corresponds to a theory without interaction.

Let us first make in (17) a change of basis

$$X_\mu = \partial_\mu - \alpha^a{}_\mu X_a \tag{18}$$

imposing that it preserves the direct-product character. It leads to

$$\begin{aligned} [X_\mu, X_\nu] &= -\beta^a{}_{\mu\nu} X_a \\ [X_a, X_\mu] &= 0 \\ [X_a, X_b] &= f^c{}_{ab} X_c \end{aligned} \tag{19}$$

It follows from (14) that

$$X_a(\alpha^b{}_\mu) = f^b{}_{ca} \alpha^c{}_\mu \tag{20}$$

This behavior characterizes α as a connection, or an adjoint-behaved 1-form. It may seem that a derivative, vacuum term is missing, but we are working on the bundle and the vacuum term only comes out when the connection is pulled back to spacetime by a section.

From (15) and (16) we obtain the expression for the nonlinearity indicator:

$$\beta^a{}_{\mu\nu} = \partial_\mu \alpha^a{}_\nu - \partial_\nu \alpha^a{}_\mu + f^a{}_{bc} \alpha^b{}_\mu \alpha^c{}_\nu \tag{21}$$

Since in a direct product the extension is central we must have

$$[X_a, \beta^c{}_{\mu\nu} X_c] = 0 \tag{22}$$

and consequently

$$X_a(\beta^c{}_{\mu\nu}) = f^c{}_{ba} \beta^b{}_{\mu\nu} \tag{23}$$

This condition, which can be equally obtained from (7), says that also β

belongs to the adjoint representation of the group whose generators are represented by X_a.

The above algebraic configuration is just the structure appearing in a gauge theory, where β is the field strength of the gauge potential α. The change of basis (18) corresponds to the covariant derivative introduced in gauge theories by the minimal coupling prescription.

Gauge field dynamics can be obtained via the duality prescription: the sourceless field equations are written just like the Bianchi identities, but applied to the dual of the field strength. This dual depends on the metric. Recall that, of Maxwell's equations, one pair is metric insensitive (they are Bianchi identities), whereas the other is metric dependent (they are the real dynamical equations). In principle, any metric which is preserved by the derivation will do, but different metrics lead to inequivalent equations. We simply assume the existence of such a metric. We obtain the field equations for α by first finding the Jacobi identity for three fields X_μ, X_ν, X_ρ in algebra (19)—which gives a Bianchi identity—and then applying the duality prescription. The Yang–Mills equations come out:

$$X_\mu \beta^{a\mu\nu} = 0 \tag{24}$$

From the point of view of the theory of algebra extensions, the next natural step would be to break the direct product in (19) by another change of basis,

$$X'_\mu = X_\mu - \gamma^a{}_\mu X_a \tag{25}$$

and investigate the kind of physical theory to which the resulting configuration can be associated. Expression (25) leads to the following commutation relations:

$$\begin{aligned} [X'_\mu, X'_\nu] &= -\beta'^a{}_{\mu\nu} X_a \\ [X'_\mu, X_a] &= C'^c{}_{\mu a} X_c \\ [X_a, X_b] &= f^c{}_{ab} X_c \end{aligned} \tag{26}$$

which just corresponds to the extended theory of the previous section. Now, it follows from (14) that

$$X_b(\gamma^a{}_\mu) = f^a{}_{cb}\gamma^c{}_\mu + C'^a{}_{\mu b} \tag{27}$$

Comparison with (20) shows that $C'^a{}_{\mu b}$ measures the deviation from covariant behavior of the object $\gamma^a{}_\mu$ appearing in (25). With the help of (18), we can express (25) as

$$X'_\mu = \partial_\mu - \sigma^a{}_\mu X_a \tag{28}$$

with $\sigma^a{}_\mu \equiv (\alpha^a{}_\mu + \gamma^a{}_\mu)$. We shall call (28) a *generalized derivative*. In fact we shall, from now on, give that name to each derivative which is not the standard gauge-covariant derivative. The behavior of σ under the group action is

$$X_b(\sigma^a{}_\mu) = f^a{}_{cb}\sigma^c{}_\mu + C'^a{}_{\mu b} \tag{29}$$

The new nonlinearity indicator $\beta'^a{}_{\mu\nu}$ can be obtained from (15) using (16) and (21):

$$\beta'^a{}_{\mu\nu} = \partial_\mu\sigma^a{}_\nu - \partial_\nu\sigma^a{}_\mu + f^a{}_{bc}\sigma^b{}_\mu\sigma^c{}_\nu - C'^a{}_{c\mu}\sigma^c{}_\nu + C'^a{}_{c\nu}\sigma^c{}_\mu \tag{30}$$

($C'^a{}_{b\mu} = -C'^a{}_{\mu b}$). This is the general expression for the deviation from linearity in the presence of an object with behavior given by (29). The behavior of β' under the group action is fixed by the Jacobi identity (7), replacing Y_μ by X'_μ and C by C'.

The dynamics associated to algebra (26) is obtained by applying the duality prescription in a way analogous to that leading to the Yang–Mills equations. The Jacobi identity involving three fields X'_μ in (26) is

$$\begin{aligned} X'_\mu(\beta'^a{}_{\nu\sigma}) - C'^a{}_{c\mu}\beta'^c{}_{\nu\sigma} + X'_\sigma(\beta'^a{}_{\mu\nu}) - C'^a{}_{c\sigma}\beta'^c{}_{\mu\nu} \\ + X'_\nu(\beta'^a{}_{\sigma\mu}) - C'^a{}_{c\nu}\beta'^c{}_{\sigma\mu} = 0 \end{aligned} \tag{31}$$

Applying this expression to the dual of $\beta'^a{}_{\mu\nu}$, we find for the field equations

$$X'_\mu\beta'^{a\mu\nu} - C'^a{}_{d\mu}\beta'^{d\mu\nu} = 0 \tag{32}$$

These equation are, of course, linked to the choice of C', which is constrained by the Jacobi identity (8).

The set of commutators (26) can be obtained directly from (17) by the basis change (28). The above two-step procedure is, however, appropriate to show how it can be attained from the algebraic scheme of a gauge theory. The 1-form $\sigma^a{}_\mu$ appearing in the generalized derivative can be seen as a connection deformed by the addition of a noncovariant form (Aldrovandi, 1991b).

We can infer using (30) in (32) that a mass term for σ can appear. Thus, this second change of basis (or a change of basis in a gauge configuration) leads to a theory with massive vector fields which do not behave like connections. This is what happens in the Weinberg–Salam model.

Another change of basis may be introduced as follows. Going back to (31), we see that it has the form of a Bianchi identity for a still more general, enlarged derivative X'^*_μ, which can be defined by its action on a indexed object Z^c as

$$X'^*_\mu(Z^c) = X'_\mu(Z^c) - C'^c{}_{a\mu}(Z^a) \tag{33}$$

To be acceptable as a derivative, X'^*_μ must obey the Leibniz rule, which leads to some interesting consequences. For example, for a scalar of type Z^aZ_a,

$$X'^*_\mu(Z^a Z_a) = X'_\mu(Z^a Z_a) \tag{34}$$

For a lower-indexed object,

$$X'^*_\mu(Z_c) = X'_\mu(Z_c) + C'^e{}_{c\mu} Z_e \tag{35}$$

and for a mixed product,

$$X'^*_\mu(Z^d J_c) = X'_\mu(Z^d J_c) + C'^e{}_{c\mu}(Z^d J_e) - C'^d{}_{e\mu}(Z^e J_c) \tag{36}$$

Expression (33) leads to the commutators

$$[X'^*_\mu, X'^*_\nu](Z^c) = -\beta'^a{}_{\mu\nu} X_a(Z^c) - R'^c{}_{a\mu\nu} Z^a \tag{37}$$

$$[X'^*_\mu, X_a](Z^c) = X_a(C'^c{}_{d\mu}) Z^d \tag{38}$$

where $\beta'^a{}_{\mu\nu}$ is given by (30) and

$$R'^c{}_{a\mu\nu} = X'_\mu C'^c{}_{a\nu} - X'_\nu C'^c{}_{a\mu} - C'^c{}_{b\mu} C'^b{}_{a\nu} + C'^c{}_{b\nu} C'^b{}_{a\mu} \tag{39}$$

The relation between C' and its algebraic derivative is given by Jacobi identity (8).

Besides the same nonlinear coefficient $\beta'^a{}_{\mu\nu}$ appearing in (26), the extra nonlinear term $R'^c{}_{\alpha\mu\nu}$ appears. The relationship between these coefficients is provided by the Jacobi identity for the fields X_a, X'_μ, X'_ν:

$$X_b(\beta'^a{}_{\mu\nu}) + f^a{}_{bc}\beta'^c{}_{\mu\nu} + R'^a{}_{b\mu\nu} = 0 \tag{40}$$

The dynamics corresponding to configuration (37) and (38) is examined in the next two sections.

4. ENLARGING THE GEOMETRY

In the fiber bundle structure, a local basis always exists (Cho, 1975) in which the commutation table takes the form (19). This means that real geometry, or real bundles, only admit quantities behaving as connections. Extended field algebras involve an object behaving differently. We endeavor now to move a little beyond the strictly geometric canvas by finding which properties can still be retained in the presence of such a misbehaving element and which requirements should be imposed if we insist on remaining as near as possible to usual geometry.

First, we would like to relate the new objects to gravitation, and $R'^c{}_{a\mu\nu}$ would bear some resemblance to a curvature written in the basis $\{X'_\nu\}$ *if* C' were a connection. However, (39) is not the correct expression for a curvature. A term involving a contraction of the basis anholonomy coefficient with the connection is missing. Furthermore, the $\beta'^a{}_{\mu\nu}$ term in (37) should be a torsion, or an anholonomy, but is not: for that, it should have values along X'^*_μ.

Under the assumption that the dimensions of the two algebras are the same, a solution to these problems comes from the following considerations. The vector spaces underlying two algebras of the same finite dimension are isomorphic (we insist: only as vector spaces). The isomorphism can be realized by a mapping H between them, such as

$$X'_\mu = H^a{}_\mu X_a \tag{41}$$

The mapping described by $H^a{}_\mu$ should be invertible. If we have spacetime in mind, the group should be itself 4-dimensional. The isomorphism in view would actually be between the tangent spaces, and should hold at each point of the manifold. Provided some reasonable differentiability conditions are met, the set $\{H^a{}_\mu\}$ will be similar to a tetrad field. We shall use for the inverse the usual tetrad notation, so that $H^a{}_\mu H_b{}^\mu = \delta^a_b$ and $H^a{}_\mu H_a{}^\nu = \delta^\nu_\mu$. Applying (41) to the second commutator in (26), we obtain

$$X_a H^d{}_\mu = f^d{}_{ca} H^c{}_\mu - C'^d{}_{\mu a} \tag{42}$$

A brief calculation leads to

$$[X'_\mu, X'_\nu] = -\beta'^\rho{}_{\mu\nu} X'_\rho \tag{43}$$

with

$$\beta'^\rho{}_{\mu\nu} = \beta'^a{}_{\mu\nu} H_a{}^\rho \tag{44}$$

showing $(-\beta'^\rho{}_{\mu\nu})$ in the role of the nonholonomy coefficient for the basis $\{X'_\mu\}$.

Taking (41) into (33), we obtain the relation between X_a and X'^*_μ:

$$X_a Z^c = H^\mu{}_a (X'^*_\mu Z^c + C'^c{}_{b\mu} Z^b) \tag{45}$$

The commutator (37) can then be rewritten as

$$[X'^*_\mu, X'^*_\nu](Z^c) = -\beta'^\rho{}_{\mu\nu} X'^*_\rho Z^c - \mathcal{R}^c{}_{a\mu\nu} Z^a \tag{46}$$

where now

$$\mathcal{R}^c{}_{a\mu\nu} = X'_\mu C'^c{}_{a\nu} - X'_\nu C'^c{}_{a\mu} - C'^c{}_{b\mu} C'^b{}_{a\nu} + C'^c{}_{b\nu} C'^b{}_{a\mu} + \beta'^\rho{}_{\mu\nu} C'^c{}_{a\rho} \tag{47}$$

This is the correct expression of the curvature of a connection C' in basis $\{X'_\mu\}$ (Nakahara, 1990). It can be shown that the commutator in (46), if applied to a mixed object with internal and external indices, only acts on those internal.

Let us examine some more properties of the candidate connection C'. Taking $X_a = H^\mu{}_a X'_\mu$ into the second commutator of (26), we obtain

$$C'^{b}{}_{a\mu} = H^{b}{}_{\lambda} C'^{\lambda}{}_{\nu\mu} H_{a}{}^{\nu} + H_{a}{}^{\nu} X'{}_{\mu}(H^{b}{}_{\nu}) \tag{48}$$

which shows that C' behaves, under the the action of $H^{\mu}{}_{a}$, as a connection of the linear group would behave under a tetrad, with $C'^{\lambda}{}_{\nu\mu} = \beta'^{\lambda}{}_{\mu\nu}$. A curious consequence is that

$$X'^{*}{}_{\lambda}\beta'^{\rho}{}_{\mu\nu} = X'{}_{\lambda}\beta'^{\rho}{}_{\mu\nu} \tag{49}$$

The torsion tensor is $T^{\rho}{}_{\mu\nu} = -\beta'^{\rho}{}_{\mu\nu}$. From (42) and (48) it can be written as

$$T^{\rho}{}_{\mu\nu} = H_{a}{}^{\rho}(X'{}_{\mu} H^{a}{}_{\nu} - X'{}_{\nu} H^{a}{}_{\mu} + f^{a}{}_{bc} H^{b}{}_{\mu} H^{c}{}_{\nu}) \tag{50}$$

The deformed Yang–Mills field strength acquires the rank of a torsion. Due to the last term in (50), it would be better to call $T^{\rho}{}_{\mu\nu}$ a "generalized torsion tensor." It reduces to usual torsion in the Abelian case (Aldrovandi, 1991b). One should remember that usual torsion is a 2-form with values in the algebra of translation generators. In the present case, torsion has values in the assumed gauge group algebra, which is non-Abelian. This is the origin of the extra term.

As with usual tetrads, the $H^{a}{}_{\mu}$ can be used to transmute indices from the gauge algebra to spacetime and vice versa. However, due to the presence of noncovariant objects, the usual properties do not follow automatically; every one must be verified by direct calculation. For example, computation gives, for the curvature, just what we would expect from a tensorial object,

$$\mathcal{R}^{\rho}{}_{\sigma\mu\nu} = X'{}_{\mu} C'^{\rho}{}_{\sigma\nu} - X'{}_{\nu} C'^{\rho}{}_{\sigma\mu} - C'^{\rho}{}_{\alpha\mu} C'^{\alpha}{}_{\sigma\nu} + C'^{\rho}{}_{\alpha\nu} C'^{\alpha}{}_{\sigma\mu} + \beta'^{\gamma}{}_{\mu\nu} C'^{\rho}{}_{\sigma\gamma} \tag{51}$$

As in (49), it happens that

$$X'^{*}{}_{\lambda}\mathcal{R}^{\rho}{}_{\sigma\mu\nu} = X'{}_{\lambda}\mathcal{R}^{\rho}{}_{\sigma\mu\nu} \tag{52}$$

It is important to notice that the enlarged derivative, acting on objects with the indices transmuted to spacetime indices, changes its form. As happens with covariant derivatives, it will take a different aspect when acting on objects with one or two indices. The simplest way to discover its form is to read it from the Bianchi identities, as we shall do below.

5. APPROACHING A GRAVITATIONAL MODEL

In the previous section we obtained (i) an anholonomy or torsion term in the commutator and (ii) the correct expression for the curvature in a nonholonomic basis. In what follows we show that two other geometrical landmarks also hold: the two Bianchi identities for linear connections. Despite their purely geometrical character, Bianchi identities are, both in gauge theories and in general relativity, intimately related to dynamics, so that we shall also comment on the field equations. The procedure adopted here parallels those theories. The field equations are obtained by applying the duality prescription to the sole Bianchi identity present in the gauge sector. In the gravity sector, a contracted Bianchi identity is used to recognize which expression is to be identified to the source current. Properties (49) and (52) hold in general when we derive objects with external indices only. Thus, the metric $g_{\alpha\beta}$ used in (32) can be used in the following. We see from (41) that, if preserved by X'^*_μ, it also will be gauge invariant. Recognizing (33) in the field equation (32) and adding a source current, we arrive at

$$X'^*_\mu \beta'^{a\mu\nu} = J^{a\nu} \tag{53}$$

As the deformed Yang–Mills field coincides with torsion, this equation fixes the dynamics for both. Applying X'^*_ν to this equation, we find a rather surprising result:

$$X'^*_\nu J^{a\nu} = 0 \tag{54}$$

This "current conservation" shows that some invariance must still be at work, although its meaning is not clear. Notice that the commutation relations, the new covariant derivatives and the dynamics of $\sigma^c{}_\mu$ have all been constructed or obtained with respect to the Jacobi identities, which are, for tangent vector fields, integrability conditions. Once the duality symmetry also is supposed to hold, some invariance is to be expected.

Which kind of gravitational model would turn up? Using (47), we can write Eq. (40) as

$$X_a \beta'^c{}_{\mu\nu} + f^c{}_{ea}\beta'^e{}_{\mu\nu} = -\mathcal{R}^c{}_{a\mu\nu} + H^\rho{}_d \beta'^d{}_{\mu\nu} C'^c{}_{a\rho} \tag{55}$$

which presents $\mathcal{R}^b{}_{a\mu\nu}$ as an effect of β's noncovariance. Applying $H^\alpha{}_c H^a{}_\sigma$, we find

$$\mathcal{R}^\alpha{}_{\sigma\mu\nu} + X'^*_\sigma(\beta'^\alpha{}_{\mu\nu}) = 0 \tag{56}$$

The Ricci tensor is not symmetric,

$$\mathcal{R}_{\sigma\nu} + X'^*_\sigma(\beta'^\alpha{}_{\alpha\nu}) = 0 \tag{57}$$

which is to be expected in the presence of torsion. The gravitational sector would be close to an Einstein–Cartan model, but with dynamical torsion. Combined with (56), (51) leads to

$$\begin{aligned} &X'_\lambda(\beta'^\rho{}_{\nu\mu}) + X'_\mu(\beta'^\rho{}_{\lambda\nu}) + X'_\nu(\beta'^\rho{}_{\mu\lambda}) \\ &\quad = -\beta'^\alpha{}_{\mu\nu}\beta'^\rho{}_{\lambda\alpha} - \beta'^\alpha{}_{\lambda\mu}\beta'^\rho{}_{\nu\alpha} - \beta'^\alpha{}_{\nu\lambda}\beta'^\rho{}_{\mu\alpha} \end{aligned} \tag{58}$$

Let us now show that the two Bianchi identities have the same formal aspect they have in usual geometry. We start by calculating the Jacobi identity for X'^*_μ, X'^*_ν, and X'^*_λ,

$$[X'^*_\lambda, [X'^*_\mu, X'^*_\nu]](Z^c) + [X'^*_\nu, [X'^*_\lambda, X'^*_\mu]](Z^c) + [X'^*_\mu, [X'^*_\nu, X'^*_\lambda]](Z^c) = 0 \tag{59}$$

We first obtain one of the three cyclic terms:

$$[[X'^*_\lambda, [X'^*_\mu, X'^*_\nu]](Z^c) = [\beta'^\rho{}_{\mu\nu}\beta'^\alpha{}_{\lambda\rho} - X'^*_\lambda\beta'^\alpha{}_{\mu\nu}]X'^*_\alpha(Z^c) - [X'^*_\lambda\mathcal{R}^c{}_{a\mu\nu} - \beta'^\rho{}_{\mu\nu}\mathcal{R}^c{}_{a\rho\lambda}]Z^a$$

We can here read the enlarged derivative acting on an object with one transmuted index. The expression inside the first bracket in the right-hand-side is, up to the sign, equal to $X'_\lambda\beta'^\alpha{}_{\mu\nu} - C'^\alpha{}_{\rho\lambda}\beta'^\rho{}_{\mu\nu}$. This is the enlarged derivative acting on $\beta'^\alpha{}_{\mu\nu}$.

Applying (56), the tensorial character of $\mathcal{R}$, and $X'^*_\lambda(H_a{}^\sigma) = H_a{}^\delta C'^\sigma{}_{\delta\lambda}$, we can rewrite this expression as

$$[X'^*_\lambda, [X'^*_\mu, X'^*_\nu]](Z^c) = [\beta'^\rho{}_{\mu\nu}\beta'^\alpha{}_{\lambda\rho} + \mathcal{R}^\alpha{}_{\lambda\mu\nu}]X'^*_\alpha(Z^c) + H^c{}_\alpha H_a{}^\sigma[\mathcal{R}^\rho{}_{\sigma\mu\nu}\beta'^\alpha{}_{\lambda\rho} - \mathcal{R}^\alpha{}_{\rho\mu\nu}\beta'^\rho{}_{\lambda\sigma} - \mathcal{R}^\alpha{}_{\sigma\rho\lambda}\beta'^\rho{}_{\mu\nu} - X'^*_\lambda\mathcal{R}^\alpha{}_{\sigma\mu\nu}]Z^a$$

The identity (59) becomes then

$$\begin{aligned} &X'^*_\alpha(Z^c)[\beta'^\rho{}_{\mu\nu}\beta'^\alpha{}_{\lambda\rho} + \beta'^\rho{}_{\lambda\mu}\beta'^\alpha{}_{\nu\rho} + \beta'^\rho{}_{\nu\lambda}\beta'^\alpha{}_{\mu\rho} + \mathcal{R}^\alpha{}_{\lambda\mu\nu} + \mathcal{R}^\alpha{}_{\nu\lambda\mu} \\ &\quad + \mathcal{R}^\alpha{}_{\mu\nu\lambda}] = -H^c{}_\alpha H_a{}^\sigma Z^a\{X'^*_\nu\mathcal{R}^\alpha{}_{\sigma\lambda\mu} - C'^\alpha{}_{\rho\nu}\mathcal{R}^\rho{}_{\sigma\lambda\mu} \\ &\quad + C'^\rho{}_{\sigma\nu}\mathcal{R}^\alpha{}_{\rho\lambda\mu} - \beta'^\rho{}_{\lambda\nu}\mathcal{R}^\alpha{}_{\sigma\rho\mu} + X'^*_\lambda\mathcal{R}^\alpha{}_{\sigma\mu\nu} - C'^\alpha{}_{\rho\lambda}\mathcal{R}^\rho{}_{\sigma\mu\nu} \\ &\quad + C'^\rho{}_{\sigma\lambda}\mathcal{R}^\alpha{}_{\rho\mu\nu} - \beta'^\rho{}_{\mu\lambda}\mathcal{R}^\alpha{}_{\sigma\rho\nu} + X'^*_\mu\mathcal{R}^\alpha{}_{\sigma\nu\lambda} - C'^\alpha{}_{\rho\mu}\mathcal{R}^\rho{}_{\sigma\nu\lambda} \\ &\quad + C'^\rho{}_{\sigma\mu}\mathcal{R}^\alpha{}_{\rho\nu\lambda} - \beta'^\rho{}_{\mu\nu}\mathcal{R}^\alpha{}_{\sigma\rho\lambda}\} = 0 \end{aligned} \tag{60}$$

With the term proportional to $X'^*_\alpha(Z^c)$ in mind, we calculate

$$\begin{aligned} &\mathcal{R}^\alpha{}_{\lambda\mu\nu} + \mathcal{R}^\alpha{}_{\nu\lambda} + \mathcal{R}^\alpha{}_{\nu\lambda\mu} + \beta'^\rho{}_{\mu\nu}\beta'^\alpha{}_{\lambda\rho} + \beta'^\rho{}_{\lambda\mu}\beta'^\alpha{}_{\nu\rho} + \beta'^\rho{}_{\nu\lambda}\beta'^\alpha{}_{\mu\rho} \\ &\quad = 2[X'_\mu C'^\alpha{}_{\lambda\nu} + X'_\nu C'^\alpha{}_{\mu\lambda} + X'_\lambda C'^\alpha{}_{\nu\mu} - \beta'^\rho{}_{\mu\nu}\beta'^\alpha{}_{\lambda\rho} \\ &\qquad - \beta'^\rho{}_{\lambda\mu}\beta'^\alpha{}_{\nu\rho} - \beta'^\rho{}_{\nu\lambda}\beta'^\alpha{}_{\mu\rho}] \end{aligned}$$

The term inside the brackets vanishes by the Jacobi identity (31) with all the indices in spacetime. The left-hand side is the factor of $X'^*_\alpha(Z^c)$ in the first term of (60), which consequently vanishes, too. The remaining content of (60) is the vanishing of the term proportional to Z^a, whose meaning we examine in the following. Notice that the above left-hand side has another interest: the fact that it is zero, combined with (58), results in

$$\mathcal{R}^\alpha{}_{\lambda\mu\nu} + \mathcal{R}^\alpha{}_{\nu\lambda\mu} + \mathcal{R}^\alpha{}_{\mu\nu\lambda} = X'_\lambda(\beta'^\alpha{}_{\nu\mu}) + X'_\mu(\beta'^\alpha{}_{\lambda\nu}) + X'_\nu(\beta'^\alpha{}_{\mu\lambda}) \tag{61}$$

As $\beta'^\alpha{}_{\mu\nu}$ is the torsion, this is just the expression for the first Bianchi identity to which linear connections submit (Kobayashi and Nomizu, 1963).

To analyze the vanishing of the term proportional to Z^a, it is convenient to read in (60) itself the form of the enlarged derivative in the nonholonomic basis $\{X'_\mu\}$ when applied to an object with two transmuted indices, like $\mathcal{R}^\alpha{}_{\lambda\mu\nu}$. It has the same expression as the usual covariant derivative:

$$D_\nu \mathcal{R}^\alpha{}_{\sigma\lambda\mu} = X'_\nu \mathcal{R}^\alpha{}_{\sigma\lambda\mu} - C'^\alpha{}_{\rho\nu} \mathcal{R}^\rho{}_{\sigma\lambda\mu} + C'^\rho{}_{\sigma\nu} \mathcal{R}^\alpha{}_{\rho\lambda\mu} - \beta'^\rho{}_{\lambda\nu} \mathcal{R}^\alpha{}_{\sigma\rho\mu} \tag{62}$$

where use has been made of (52). The identity then reduces to

$$D_\nu \mathcal{R}^\alpha{}_{\sigma\lambda\mu} + D_\lambda \mathcal{R}^\alpha{}_{\sigma\mu\nu} + D_\mu \mathcal{R}^\alpha{}_{\sigma\nu\lambda} = 0 \tag{63}$$

which has the form of the second Bianchi identity.

We now follow a path which parallels that used in general relativity to identify the geometrical object appearing in the field equation, analogous to the Einstein tensor. Contracting first α with λ and then using the preserved metric $g_{\alpha\beta}$ to contract the remaining indices, we get

$$D_\nu \mathcal{R}^\mu{}_\mu + D_\alpha \mathcal{R}^{\alpha\mu}{}_{\mu\nu} + D_\mu \mathcal{R}^\mu{}_\nu = 0$$

This contracted Bianchi identity takes the form

$$D_\alpha G^{\alpha\sigma} = 0 \tag{64}$$

provided we define

$$G^{\alpha\sigma} = \mathcal{R}^{\alpha\sigma} - g^{\alpha\sigma}\mathcal{R} - g^{\sigma\nu}\mathcal{R}^{\alpha\mu}{}_{\mu\nu} \tag{65}$$

This expression would lead to an object quite similar to the Einstein tensor if $\mathcal{R}^{\alpha\beta}{}_{\lambda\mu}$ were antisymmetric in the first two indices.

We have obtained the two Bianchi identities of usual geometry with torsion. To recover all the features of a real geometry the only missing point is the direct product of the vector field algebras. We see in (38) the possibility of recovering the direct product by setting $X_a(C'^c{}_{d\mu})Z^d = 0$, which includes the invariance of C' under the group action,

$$X_a(C'^c{}_{d\mu}) = 0 \tag{66}$$

This would mean a constant $C'^a{}_{b\mu}$, but not a constant $C'^\rho{}_{\mu\nu}$, so that the curvature would keep its general form (51). It is important to notice that such a condition to establish the direct product could only be realized because we made the change of basis (33). It could not be done inside the extended gauge theory since we wanted to preserve the misbehaving elements.

The validity of the scheme is restricted to gauge groups with the dimension of spacetime. In consequence the extended gauge scheme, besides describing a theory with massive fields that do not behave as connections, also describes a theory for a group with the dimension of spacetime. If we take this dimension equal to 4, a group that could be chosen is the $SU(2) \otimes U(1)$ of the Weinberg–Salam model. The condition on the dimension of the group guarantees the existence of the mapping $H_a{}^\nu$ and its inversibility. The introduction of H allowed us to recover the usual geometric interpretation of curvature and torsion. Significantly, the behavior of C' is the same as that of an external connection. The gravitational sector would exhibit curvature and see the deformed gauge field as a torsion, with dynamics given, respectively, by (57) and (53).

6. OPEN QUESTIONS AND FINAL REMARKS

We have seen how, when a gauge potential ceases to behave like a connection, the bundle picture of gauge theories becomes shaky. The arena appropriate to discuss the new situation is no longer bundle theory, but the theory of Lie algebra extensions, which allows for the modified local commutators coming to the fore. The object measuring the breaking of the bundle scheme is reminiscent of a linear, external, spacetime connection. A new, noncovariant, generalized derivative emerges naturally which is analogous to that appearing in electroweak theory in the presence of a gravitational field. This suggests a link between electroweak interactions and gravitation. The suggestion is strengthened by a dimensional coincidence: spacetime and the Lie algebra of the electroweak group are both 4-dimensional and, as vector spaces, isomorphic. This isomorphism can be realized by a tetrad-like field H which, once introduced, reorganizes the whole picture. Objects corresponding to the curvature and the torsion of the candidate linear connection turn up at the right places in the commutation relations and obey formally the two Bianchi identities of differential geometry. The broken gauge field strength appears in the role of torsion. The dynamical equations obtained correspond formally to a broken gauge model on a spacetime endowed with curvature.

We are far from having solved all the questions raised by the approach. The crucial, obvious problem which remains unsolved is that of the necessary index transmutation. We do obtain quantities *resembling* a connection, a curvature, and a torsion by their behavior. The connections related to gravitation are, however, related to the Lorentz group. This means that, instead of our internal indices, we should have indices related to some vector or tensor representation of the Lorentz group. This is clear in the case of real tetrad fields, which are Lorentz vectors. Our Latin indices should be somehow transformed into Lorentz indices before we can really speak of gravitation. This question is not easy to answer in a satisfactory way. What we can do now is to speculate on possible origins for such a transmutation. A first point to look at is the definition of H. We assumed equal dimensions to avoid an ill-defined mapping and this led to the transformation (48) of C'; which takes an object with internal indices into an object exclusively "external." But the fact remains that the original group has nothing to do with spacetime. When the gauge group is the group of spacetime translations $T^{3,1}$, H reduces to the usual vierbeine fields, which appear quite naturally (Aldrovandi, 1991b). In that case C' turns up as a true connection for the linear or Lorentz group, with a Riemann curvature and an additional torsion. However, translation generators are Lorentz vectors and, in a sense, "external" from the start. We mention in passing that the Lorentz group does not affect spacetime directly, but through a representation, the vector representation in this case. It could happen that the group supposed above, say, the group of the electroweak interactions, would do the same, so that the relationship to spacetime would be realized through an intermediate, "interface" representation. This will depend on the available representations of the gauge group. The group of electroweak interactions is under study.

A point worth mentioning concerns universality. It is true that gravitation is the only universal interaction. However, the electroweak interaction presents a large amount of universality. Though with different strengths, all particles (except possibly the gluons) couple to it.

For the time being, the only positive clue we have to the possibility of transmutation is the appearance of torsion in expressions like (46). Torsion is specifically external, an effect of soldering which is absent in purely internal gauge bundles. Even when it vanishes, it is responsible for the presence of two, instead of only one, Bianchi identities. Another point worth remembering is that our approach is, up to now, purely classical. It is possible that transmutation come as a quantum effect. Indeed, getting "spin from isospin" was studied in the seventies (Jackiw and Rebbi, 1976; Hasenfratz and 't Hooft, 1976; Goldhaber, 1976) as an instanton-induced transmutation of exactly the required kind. What we have done here has been to leave this question aside and investigate the purely formal aspects of the approach, to see whether it presents points enticing enough to justify further study. We think the results are highly positive.

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